SI112: Advanced Geometry	Spring 2018
Lecture Note 2 — Mar. 1st, Thursday	
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1 Lecture 2

"*Only if it is open!*"

— who said it

1.1 Overview of This Lecture

After introducing Zorn's lemma and Axiom of Choice (see lecture note 1), we dive straight into the world of continuity, where ϵ and δ live. Following Meldelson, the concept of *continuity* is defined, and refined, with increasing abstraction. That is, we shortly review the real-valued continuous functions, and then define continuity on metric space. As the concepts (e.g., open ball, neighborhood) defined, the theorems get refined and become more and more abstract. Thankfully, all definitions and theorems devoloped in this lecture can be visualized and you can therefore see the geometric intuition behind the complicated manipulation of ϵ and δ.

1.2 Proof of Things

Definition 1.2.1 (Metric Space)**.** Metric Space is a set X together with a function $d : X \times X \to \mathbb{R}$ satisfying 1) $d(x, y) = 0 \iff x = y$, 2) $d(x, y) = d(y, x), \forall x, y \in X$, and 3) $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.2.2 (norm on $\mathbb{R}^n$)**.** norm on $\mathbb{R}^n$ is $|| \cdot || : \mathbb{R}^n \to \mathbb{R}$ satisfying 1) $||x|| = 0 \iff x = 0$, 2) $||\alpha x|| = |\alpha| \cdot ||x||, \forall \alpha \in \mathbb{R}$, and 3) $||x + y|| \leq ||x|| + ||y||$.

You may consider property (3) of both norm $||\cdot||$ and metric d to be triangular inequality.

Example 1.2.3 (examples of norm)**.** 1) ℓ_2 norm: $||x||_2 = (\Sigma_{i=1}^n x_i^2)^{\frac{1}{2}}$, 2) ℓ_1 norm: $||x||_1 = (\Sigma_{i=1}^n |x_i|$, and 3) ℓ_∞ norm: $||x||_\infty = \max_i |x_i|$. See also Vector Norm and Matrix Norm in wikipedia.

Proposition 1.2.4. *Let $||\cdot||$ be a norm on $\mathbb{R}^n$. Then $(\mathbb{R}, d)$ is a metric space where $d(x, y) = ||x - y||$.*

proof skeleton. proving this proposition will get you familiar with the definitions above. Try it. □

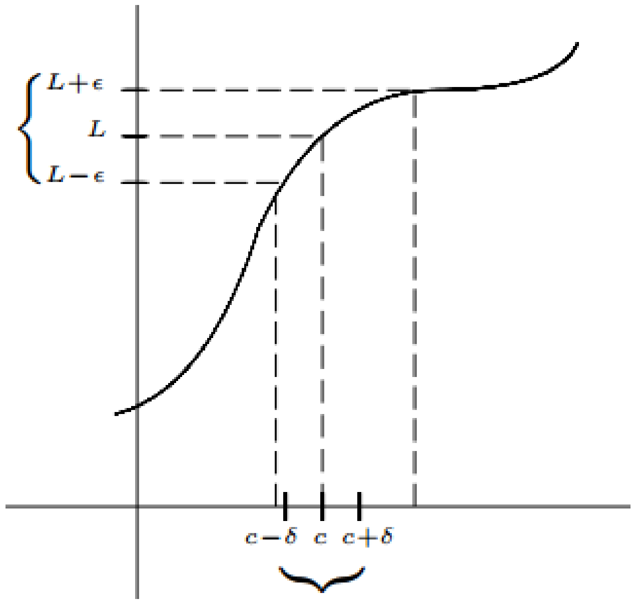

Figure 1: A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous* at a point $c \in \mathbb{R}$, if given $\epsilon > 0$, there is a $\delta > 0$, such that $|f(x) - f(c)| < \epsilon$ ($L = f(c)$ in the figure), whenever $|x - c| < \delta$. The function f is said to be *continuous* if it is continuous at each point of $\mathbb{R}$

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Before learning the definition of continuity on metric space, you may want to review real-valued continuous functions which you've learned before.

Definition 1.2.5 (real-valued continuous function)**.** This is definition 3.1, chapter 2 in Medelson. See also figure 1.

Definition 1.2.6 (continuity on metric space, v.1)**.** Let (X, d) and (Y, d') be metric spaces. The function $f : X \to Y$ is said to be *continuous* at the point $\alpha \in X$, if for each $\epsilon > 0$, there exists $\delta_\epsilon > 0$ satisfying

$$d(x, \alpha) < \delta_\epsilon \Rightarrow d(f(x), f(\alpha)) < \epsilon. \tag{1}$$

Exercise 1.2.7. Try to find an error in the above definition (continuity on metric space, v.1).

Exercise 1.2.8. Try to compare two definitions above. Describe their differences in your mind.

Once we know the definition of continuity on metric space, we are ready to prove the continuity of some simple functions.

Theorem 1.2.9 (theorem 3.3, chapter 2 in Mendelson)**.** *Let* (X, d) *and* (Y, d') *be metric spaces. Let* $f : X \to Y$ *be a constant function, then* f *is continuous.*

proof skeleton. Let ϵ be given, try to find δ_ϵ satisfying the definition of continuity. $\square$

Theorem 1.2.10 (theorem 3.4, chapter 2 in Medelson)**.** *Let* (X, d) *be a metric space. Then the identity function* $i : X \to X$ *is continuous.*

proof skeleton. Let ϵ be given, try to find δ_ϵ satisfying the definition of continuity. $\square$

Theorem 1.2.11 (theorem 3.6, chapter 2 in Medelson)**.** *Let* $(X, d), (Y, d'), (Z, d'')$ *be metric spaces. Let* $f : X \to Y$ *be continuous at the point* $a \in X$ *and let* $g : Y \to Z$ *be continuous at the point* $f(a) \in Y$*. Then* $gf : X \to Z$ *is continuous at the point* $a \in X$*.*

proof skeleton. All you need is just patience. Step by step. Let ϵ be given, you have to find a $\delta > 0$ such that whenever $x \in X$ and $d(x, a) < \delta$, then $d''(g(f(x)), g(f(a))) < \epsilon$. $\square$

Definition 1.2.12 (open ball, definition 4.1, chapter 2)**.** $B(a; \delta)$ is called an open ball, if it contains all the points $x \in X$ in X such that $d(a, x) < \delta$.

Exercise 1.2.13. $B(0.5; 0.5) = (0, 1)$ is an open ball on the metric space $\mathbb{R}$. Try to give a example of open ball on $\mathbb{R}^2$.

Lemma 1.2.14. *Let* X*,* Y *be sets,* $f : X \to Y$ *a function, and* $S \subset X$*,* $T \subset Y$*. Then we have* $f(S) \subset T \iff S \subset f^{-1}(T)$*.*

proof skeleton. Immediate! Just follow the definition. $\square$

Theorem 1.2.15 (continuity on metric space, v.2, theorem 4.2/4.3, chapter 2)**.** *A function* $f : (X, d) \to (Y, d')$ *is continuous at a point* $a \in X$ *if and only if given* $\epsilon > 0$ *there is a* $\delta > 0$ *such that*

$$f(B(a; \delta)) \subset B(f(a); \epsilon)\,, \tag{2}$$

or

$$B(a; \delta) \subset f^{-1}(B(f(a); \epsilon))\,. \tag{3}$$

proof skeleton. If equation 2 is proved, the equation 3 is immediate because of the lemma above. Observe that the equation 2 is just the open ball version (v.2) of the definition of continuity. Try to translate the equation 1 into equation 2. $\square$

Definition 1.2.16 (neighborhood, definition 4.4, chapter 2)**.** Let (X, d) be a metric space and $a \in X$. A subset N of X is called a *neighborhood* of a if there is a δ such that $B(a; \delta) \subset N$.

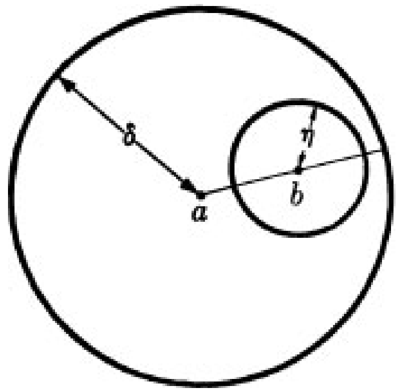

Figure 2: Choose arbitrarily a point $b \in X$, then you have to find a ball $B(b;\eta)$ contained in the ball $B(a;\delta)$.

Lemma 1.2.17. *Let (X, d) be a metric space and $a \in X$. For each $\delta > 0$, the open ball $B(a;\delta)$ is a neighborhood of each of its points.*

proof skeleton. All is in figure 2. □

Exercise 1.2.18. Let the set S be a neighborhood of a point $a \in X$. Prove that the set containing S is also a neighborhood of a.

Theorem 1.2.19 (continuity on metric space, v.3, theorem 4.6, chapter 2)**.** *Let $f : (X, d) \to (Y, d')$. f is continuous at a point $a \in X$ if and only if for each neighborhood M of $f(a)$ there is a corresponding neighborhood N of a such that*

$$f(N) \subset M, \tag{4}$$

or equivalently,

$$N \subset f^{-1}(M). \tag{5}$$

Proof. Equation 4 implies equation 5. To prove the equation 4, you need to understand all of the previous definitions and theorems. You may use the continuity theorem in terms of open ball (v.2), so you need to think about what is the connection between an open ball and the neighborhood of a point. Also think pictorially. □

Theorem 1.2.20 (continuity on metric space, v.4, theorem 4.7, chapter 2)**.** *Let $f : (X, d) \to (Y, d')$. f is continuous at a point $a \in X$ if and only if for each neighborhood M of $f(a)$, $f^{-1}(M)$ is a neighborhood of a.*

proof skeleton. This is a homework problem. Notice that this theorem is quite similar to the previous one, with only one difference. We have two variables N and M as the neighborhoods of X and Y in continuity v.3, while only one variable M is used in continuity v.4 (we use $f^{-1}(M)$ to replace N). In this sense v.4 is terser, and easier to remember. $\square$

1.3 Further Reading

Mendelson, chapter 2. Try to solve some problems at the end of chapter.

Bertrand Russell, *Logicomix*.