SI112: Advanced Geometry		Spring 2018
	Lecture Note 3 — Mar. 6th, Tuesday	
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1 Lecture 3

1.1 Overview of This Lecture

At the beginning of this lecture, we discussed the continuity in terms of neighborhoods (see lecture note 2). Then we introduced *limit* of a sequence in metric space (1.2.2), for which we reviewed limit of a sequence of real numbers (1.2.1). Of course, the concept of limit can also be described in the language of neighborhood (1.2.3). Once a new operation *limit* is defined, we want to see if this operation is consistent with the operations defined previously (e.g., Algebraic Limit Theorem). In this spirit, we arrive at theorem 1.2.4, which states the connection between continuous function and limit operation: continuous functions *preserve sequential convergence*.

1.2 Proof of Things

Definition 1.2.1 (limit of a sequence of real numbers.)**.** Let $a_1, a_2, \dots$ be a sequence of real numbers. A real number a is said to be the *limit* of the sequence $a_1, a_2, \dots$ if, given $\epsilon > 0$, there is a positive integer N such that, whenever $n > N$, $|a - a_n| < \epsilon$. In this event we shall also say that the sequence $a_1, a_2, \dots$ *converges* to a and write $\lim_n a_n = a$.

Definition 1.2.2 (limit of a sequence in a metric space)**.** Let (X, d) be a metric space. Let $a_1, a_2, \dots$ be a sequence of points of X. A point $a \in X$ is said to be *the limit of the sequence* $a_1, a_2, \dots$ if $\lim_n d(a, a_n) = 0$. Again in this event we shall say that the sequence $a_1, a_2, \dots$ *converges* to a and write $\lim_n a_n = a$.

Be careful. Try to understand $\lim_n d(a, a_n) = 0$.

Corollary 1.2.3. *Let (X, d) be a metric space and $a_1, a_2, \dots$ be a sequence of points of X. Then $\lim_n d(a, a_n) = 0$ for a point $a \in X$ if and only if for each neighborhood V of a there is an integer N such that $a_n \in V$ whenever $n > N$.*

proof skeleton. Immediate. From this corollary we can see that, if the limit of a sequence exists and N is big enough, the sequence will *eventually* fall into a neighborhood V of the limit a. Try to picture it on the real line. □

Theorem 1.2.4 (theorem 5.4, chapter 2)**.** *Let* $(X,d),(Y,d')$ *be metric spaces. A function* $f : X \to Y$ *is continuous at a point* $a \in X$ *if and only if, whenever* $\lim_n a_n = a$ *for a sequence* $a_1, a_2, \ldots$ *of points of* X*,* $\lim_n f(a_n) = f(a)$*.*

proof skeleton. This theorem says that a continuous function *preserves sequential convergence*, i.e., a convergent sequence undergoing a function/transformation is still convergent, or, i.e., $\lim_n a_n = a \Rightarrow \lim_n f(a_n) = f(a) = f(\lim_n a_n)$.

On the one hand, suppose that f is continuous at a point $a = \lim_n a_n$, and there is a sequence $a_1, a_2, \ldots$ of points of X (why we can suppose like this when proving this direction, what if there is no such sequences in X at all? *hint*: recall that how we prove $\emptyset \subset A$, where A is any set), we need to show $\lim_n f(a_n) = f(a)$. By Corollary 1.2.3, the sequence $a_1, a_2, \ldots$ will eventually fall into any specified neighborhood of a, and of course, given that V is a neighborhood of $f(a)$, it will *eventually* fall into $f^{-1}(V)$, since $f^{-1}(V)$ is a neighborhood of a (why? by which theorem?). Hence the sequence $f(a_1), f(a_2), \ldots$ will eventually fall into V (why?), which means $\lim_n f(a_n) = f(a)$ (why? by which theorem/corollary?).

On the other hand, suppose the function f preserves sequential convergence, where the sequence converges at a point a, you need to show f is continuous at a. It is proved in the lecture by proving that f is not sequential-convergence-preserving if f is not continuous. Following the lecture, you need to construct something like $(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots)$, which is not convergent. □

Recall that a open ball is a neighborhood of each of its points (see lecture note 2, or pictures, or textbooks), we define *open set*, as an abstraction of open ball, to satisfy this important property.

Definition 1.2.5. A subset O of a metric space is said to be *open* if O is a neighborhood of each of its points.

Theorem 1.2.6 (theorem 6.2, chapter 2)**.** *A subset* O *of a metric space* (X,d) *is an open set if and only if it is a union of open balls.*

proof skeleton. Try it to get familiar with open set. □

Theorem 1.2.7 (theorem 6.3, chapter 2)**.** *Let* $f : (X,d) \to (Y,d')$*. Then* f *is continuous if and only if for each open set* O *of* Y*, the subset* $f^{-1}(O)$ *is an open subset of* X*.*

proof skeleton. To prove this theorem, you are invited to think about the connection between open set and neighborhood, just like previously we invite you to think about the relationship between open ball centered at a point a and neighborhood of the point a. □

Theorem 1.2.8 (theorem 6.4, chapter 2)**.** *Let (X, d) be a metric space. Then we have*

- *The empty set $\emptyset$ is open.*
- *X is open.*
- *If $O_1, O_2, \ldots, O_n$ is open, then $O_1 \cap \cdots \cap O_n$ is open.*
- *If for each $\alpha \in I$, O_α is an open set, then $\cup_{\alpha \in I} O_\alpha$ is open.*

proof skeleton. Immediate. $\square$

1.3 Further Reading

2.5, 2.6 in Mendelson.