SI112: Advanced Geometry Spring 2018
Lecture Note 8 — Mar. 22th, Thursday
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1 Lecture 8

1.1 Overview of This Lecture

We reviewed the definition of irreducible space (1.2.1) and from it we then developed some theorems (1.2.3, 1.2.4 and 1.2.6) absent in Mendelson, for which full proofs are given. We proceeded by introducing the product of topological spaces, discussing and exploring why it is defined as it is (1.2.10, 1.2.12). Finally we played a bit (?), which ends this lecture.

1.2 Proof of Things

Definition 1.2.1 (Irreducible Space)**.** If X is not the union of two proper closed sets, i.e., there are not $Y_1, Y_2 \subsetneq X$, which are closed, such that $X = Y_1 \cup Y_2$.

Example 1.2.2. Zariski Topology on $\mathbb{R}^2$ is irreducible. Y_1, Y_2 are curves and they can not cover the entire space.

Proposition 1.2.3. *Let A, B be a subset of a topological space X, then we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

Proof. For each $x \in X$, if $x \in A \cup B$, then we have $x \in \overline{A \cup B} \iff x \in \overline{A} \cup \overline{B}$ (as you can easily verify). Hence, to prove $\overline{A \cup B} = \overline{A} \cup \overline{B}$, it is enough to show that x is a limit point of $A \cup B$ if and only if x is a limit point of A or x is a limit point of B (why?). Let $N \in \mathcal{N}_x$ denotes that N is a neighborhood of x, we finished the proof since

$$\begin{aligned} x \text{ is a limit point of } A \cup B &\iff \forall N \in \mathcal{N}_x,\ N/\{x\} \cap (A \cup B) \neq \emptyset \\ &\iff \forall N \in \mathcal{N}_x,\ (N/\{x\} \cap A) \cup (N/\{x\} \cap B) \neq \emptyset \\ &\iff \forall N \in \mathcal{N}_x,\ N/\{x\} \cap A \neq \emptyset \text{ or } N/\{x\} \cap B \neq \emptyset \\ &\iff x \text{ is a limit point of } A \text{ or } x \text{ is a limit point of } B\ . \end{aligned} \tag{1}$$

□

Theorem 1.2.4. *Let X be an irreducible space, and $O \subsetneq X$ and O is open in X. Then O is irreducible and dense.*

Proof. We will first prove that O is dense and then O irreducible.

Saying that O is dense is equivalent to saying that $\overline{O} = X$. Observing that

$$X = O \cup O^C = \overline{O} \cup O^C \text{ (why?)},$$

where O^C and $\overline{O}$ closed subset of X and O^C is proper, $X = \overline{O}$, for $\overline{O} \subsetneq X$ contradicting the fact that X is irreducible.

Let F_1, F_2 be (relatively) closed in O and $O = F_1 \cup F_2$, we want to show that $O = F_1$ or $O = F_2$, from which it will follow that O is irreducible. By the definition of relative closeness, there are closed sets Z_1, Z_2 in X such that $F_1 = O \cap Z_1, F_2 = O \cap Z_2$. Then we have

$$\begin{aligned} O = (O \cap Z_1) \cup (O \cap Z_2) = O \cap (Z_1 \cup Z_2) &\Rightarrow O \subset Z_1 \cup Z_2 \\ &\Rightarrow \overline{O} \subset Z_1 \cup Z_2 \\ &\Rightarrow X = \overline{O} \subset Z_1 \cup Z_2 \subset X \\ &\Rightarrow X = Z_1 \cup Z_2 \,, \end{aligned} \tag{2}$$

which means, by irreducibility of X,

$$\begin{aligned} X = Z_1 \text{ or } X = Z_2 &\Rightarrow O \subset Z_1 \text{ or } O \subset Z_2 \\ &\Rightarrow F_1 = O \cap Z_1 = O \text{ or } F_2 = O \cap Z_2 = O \,. \end{aligned} \tag{3}$$

We finished the proof. $\square$

Remark 1.2.5. Theorem 1.2.4 shows that, given an irreducible space X, A "smaller" set $O \subset X$ is also irreducible (and dense) if O is open. The next theorem (1.2.6), in contrast, shows that a "larger" set is irreducible if the smaller one promises to be irreducible.

Theorem 1.2.6. *If $Y \subset X$, where X is a topological space, and Y is irreducible, then $\overline{Y}$ is irreducible.*

Proof. Similar to proving the irreducibility in theorem 1.2.4, given that $Z_1 \cap \overline{Y}$ and $Z_2 \cap \overline{Y}$ are closed in $\overline{Y}$, where Z_1 and Z_2 are closed in X, and $\overline{Y} = (Z_1 \cap \overline{Y}) \cup (Z_2 \cap \overline{Y}) = (Z_1 \cup Z_2) \cap \overline{Y}$, we need to show that $\overline{Y} = (Z_1 \cap \overline{Y})$ or $\overline{Y} = (Z_2 \cap \overline{Y})$.

From $\overline{Y} = (Z_1 \cup Z_2) \cap \overline{Y}$ we have

$$\begin{aligned} \overline{Y} \subset Z_1 \cup Z_2 &\Rightarrow Y \subset Z_1 \cup Z_2 \\ &\Rightarrow Y = (Z_1 \cup Z_2) \cap Y = (Z_1 \cap Y) \cup (Z_2 \cap Y) \,, \end{aligned} \tag{4}$$

which, by irreducibility of Y, means that

$$\begin{aligned} Y = Z_1 \cap Y \text{ or } Y = Z_2 \cap Y &\iff Y \subset Z_1 \text{ or } Y \subset Z_2 \\ &\iff \overline{Y} \subset Z_1 \text{ or } \overline{Y} \subset Z_2 \\ &\iff \overline{Y} = Z_1 \cap \overline{Y} \text{ or } \overline{Y} = Z_2 \cap \overline{Y} \,. \end{aligned} \tag{5}$$

We finished the proof.

□

Example 1.2.7 (Zariski topology on $\mathbb{R}^2$)**.** Let $(0,0) = \mathbb{Z}(x^2 + y^2) = \mathbb{Z}(x, y)$ (This is Punctured Plane).

Exercise 1.2.8 (Zariski topology on $\mathbb{R}^2$)**.** Let $Y = \mathbb{Z}(y - x^2)$. O is open in Y. How does it look like (check the pictures on the board, i.e., have a look at the board in the picture)?

Lemma 1.2.9 (lemma 3.7.1)**.** *Let $\mathcal{B}$ be a collection of subsets of a set X with the property that $\emptyset \in \mathcal{B}, X \in \mathcal{B}$, and finite intersection of elements of $\mathcal{B}$ is again in $\mathcal{B}$. Then the collection $\mathcal{J}$ of all subsets of X which are unions of elements of $\mathcal{B}$ is a topology.*

proof skeleton. Omitted. □

Exercise 1.2.10 (wrong definition for product of topological space)**.** Let $(X_1, \mathcal{J}_1), \ldots, (X_n, \mathcal{J}_n)$ be topological spaces, and let $X = \prod_{j=1}^{n} X_j$ and $\mathcal{J} = \prod_{j=1}^{n} \mathcal{J}_j$, i.e.,

$$X = \{(x_1, x_2, \ldots, x_n) | x_i \in X_i\}, J = \{(O_1, O_2, \ldots, O_n) | O_i \in \mathcal{J}_i\}.$$

Is $(X, \mathcal{J})$ a topological space? i.e.,

1. Is it that $\emptyset \in \mathcal{J}$? Yes, it is.

2. Is it that $X \in \mathcal{J}$? Yes, it is.

3. For each $O_1, \ldots, O_n \in \mathcal{J}$, is it that $O_1 \cap \cdots \cap O_n \in \mathcal{J}$? Yes it is. You need to prove something like

$$(O_1, O_2, \ldots, O_n) \cap (O_1^{'}, O_2^{'}, \ldots, O_n^{'}) = (O_1 \cap O_1^{'}, O_n \cap O_2^{'}, \ldots, O_n \cap O_n^{'}).$$

4. Is it that (fill the gap here)? No, it isn't. Review the picture on the board.

Remark 1.2.11 (remark for exercise 1.2.10)**.** The first 3 clauses are easy to verify. the clause 4 is fundamentally the reason that (1) the union of two linear subspaces is not neccesarily a linear subspace and that (2) the union of two groups is not neccesarily a group.

Definition 1.2.12 (product of topological spaces, definition 3.7.2)**.** The topological space $(X, \mathcal{J})$, where $\mathcal{J}$ is the collection of subsets of X that are unions of sets of the form $O_1 \times O_2 \times \cdots \times O_n$, each O_i and open subset of X_i, is called *product* of the topological spaces $(X_i, \mathcal{J}_i), i = 1, 2, \ldots, n$.

Remark 1.2.13 (remark for 1.2.12)**.** $(X, \mathcal{J})$ defined in this way is indeed a topology, as you should verify (*hint:* use lemma 1.2.9).

Definition 1.2.14 (neighborhood in product topology)**.** Let $(X, \mathcal{J})$ be a product of topological spaces. A set $N \subset X$ is said to be a neighborhood of a point $x \in X$, if there is an open set $O \in \mathcal{J}$ such that $a \in O \subset N$.

Exercise 1.2.15 (neighborhood in product topology, proposition 3.7.4)**.** Prove it: In a topological space $X = \prod_{j=1}^{n} X_j$, a subset N is a neighborhood of a point $a = (a_1, a_2, \ldots, a_n) \in N$ if and only if N contains a subset of the from $N_1 \times N_2 \times \cdots \times N_n$, where each N_i is a neighborhood of a_i.

Proposition 1.2.16. *The projection map.* $p_i : \prod_{j=1}^{n} X_j \to X_i$, *is continuous.*

Proof. Easy. For each open set $O \subset X_i$, what is the inverse image of O under p_i? i.e., what's $p_i^{-1}(O)$? □

Remark 1.2.17. Let's play a little bit. X_1, X_2 metric spaces. $(X_i, d_i) \to X_i, \mathcal{J}_i \to (X, \mathcal{J})$.

Zariski topology is not metrizable.

1.3 Further Reading

3.7 in Mendelson.