SI112: Advanced Geometry	Spring 2018
Lecture 24 — May 15th, Tuesday	
Prof. Manolis	*Scribe*: Liangzu

1 Lecture 24

1.1 Overview of This Lecture

1.2 Proof of Things

Definition 1.2.1. The spectrum of a ring R, denoted by $\mathrm{Spec}(R)$, is the set of all prime ideals in R. That is

$$\mathrm{Spec}(R) = \{I \subset R : \ I \text{ is prime ideal of } A\}.$$

Proposition 1.2.2 (contraction of an ideal)**.** *Let $f : A \to B$ be a ring homomorphism and $I \subset B$ an ideal. Then the inverse image $f^{-1}(I)$ is an ideal of A, called the contraction of I to A.*

Proof. Firstly, $0 \in f^{-1}(I)$ since $f(0) \in I$. Let $a, b \in f^{-1}(I) \Rightarrow f(a), f(b) \in I$ and $c \in A$. Then

$$f(a + b) = f(a) + f(b) \in I \Rightarrow a + b \in f^{-1}(I)$$

and

$$f(ac) = f(a)f(c) \in I \Rightarrow ac \in f^{-1}(I).$$

□

Proposition 1.2.3. *Let $f : A \to B$ be a ring homomorphism and $P \subset B$ a prime ideal. Then $f^{-1}(P)$ is a prime ideal of A.*

Proof. Let $a, b \in A$ be such that $ab \in f^{-1}(P) \Rightarrow f(ab) = f(a)f(b) \in P$. Then we have either $f(a) \in P \iff a \in f^{-1}(P)$ or $f(b) \in P \iff b \in f^{-1}(P)$, which implies $f^{-1}(P)$ is prime. □

Remark 1.2.4 (remark for Propsitions 1.2.2 and 1.2.3). If $A \subset B$ and $f : A \to B$ is the inclusion mapping. Then for a prime ideal P in B, $f^{-1}(P) = P \cap A$ is prime.

Proposition 1.2.5. *Let $f : A \to B$ be a surjective ring homomorphism and $Q \subset A$ a prime ideal containing* $\mathrm{Ker}(f)$*. Then $f(Q)$ is a prime ideal of B.*

Proof. It is easy to verify that $f(Q)$ is an ideal. Let $y_1, y_2 \in B$ be such that $y_1 y_2 \in f(Q)$. Then since f is surjective, there exist $x_1, x_2 \in A$ and $x \in Q$ such that $f(x_1) = y_1, f(x_2) = y_2$ and $f(x) = y_1 y_2$. Hence

$$f(x_1 x_2 - x) = 0 \Rightarrow x_1 x_2 - x \in \mathrm{Ker}(f) \Rightarrow x_1 x_2 - x \in Q \Rightarrow x_1 x_2 \in Q.$$

This implies either x_1 or x_2 is in Q and thus either $y_1 = f(x_1)$ or $y_2 = f(x_2)$ is in $f(Q)$. □

Corollary 1.2.6. *Let A be a ring and J an ideal of A. Then for each $P \in \mathrm{Spec}(A)$ such that $P \supset J$, $\pi(P)$ is prime, where $\pi : A \to A/J$ is the canonical homomorphism.*

Proposition 1.2.7. *Let A be a ring and J an ideal of A. Then if P_1, P_2 are ideals of A such that $J \subsetneq P_1 \subsetneq P_2$, $\pi(J) \subsetneq \pi(P_1) \subsetneq \pi(P_2)$, where $\pi : A \to A/J$ is the canonical homomorphism.*

Proof. It is obvious that $\pi(J) \subsetneq \pi(P_1)$ and $\pi(P_1) \subset \pi(P_2)$. Suppose for the sake of contradiction that $\pi(P_1) = \pi(P_2)$. Let $p_2 \in P_2 \backslash P_1$, then there is $p_1 \in P_1, j \in J$ such that $p_2 - p_1 = j \iff p_2 = j + p_1$, which implies $p_2 \in P_1$ since $j \in J \subset P_1$ and P_1 is an ideal, a contradiction. □

Proposition 1.2.8. *Let A be a ring and J an ideal of A. Then*

$$\mathrm{Spec}(A/J) = \{\pi(P) : \ P \in \mathrm{Spec}(A) \text{ and } J \subset P\},$$

where $\pi : A \to A/J$ is the canonical projection that maps $a \in A$ to $a + J \in A/J$.

Proof. Let $Q \in \mathrm{Spec}(A/J)$ and let $P = \pi^{-1}(Q)$. Then by Proposition 1.2.3, $P \in \mathrm{Spec}(A)$. We have $\pi(P) = \pi(\pi^{-1}(Q)) = Q$ since π is surjective. $[0] \in Q$ since Q is an ideal. Hence $\pi^{-1}([0]) \subset \pi^{-1}(Q) \Rightarrow J \subset P$. □

Remark 1.2.9. Let $\pi : A \to A/J$ be the canonical homomorphism. Then from the discussions above, we can see that there is a one-to-one correspondence between the prime ideals containing J in A and the prime ideals in A/J. Moreover, since if $Q_1 \subsetneq Q_2$ are two ideals in A/J, then $\pi^{-1}(Q_1) \subsetneq \pi^{-1}(Q_2)$, and Proposition 1.2.7, their "order" are preserved.

Definition 1.2.10. Let A be a ring and J an ideal of A. Then the dimension of A/J, called *Krull dimension* is the supremum of the lengths of all chains $Q_l \supsetneq Q_{l-1} \supsetneq \cdots \supsetneq Q_0$ of prime ideals $Q_l, Q_{l-1}, \ldots, Q_0 \in \mathrm{Spec}(A/J)$.

Remark 1.2.11. By Corollary 1.2.6 and Proposition 1.2.7, a chain $P_l \supsetneq P_{l-1} \supsetneq \cdots \supsetneq P_0 = P$ of prime ideals properly containing an ideal P gives rise to a chain $\pi(P_l) \supsetneq \pi(P_{l-1}) \supsetneq \cdots \supsetneq \pi(P_0) = \pi(P) = \{[0]\}$. Moreover, if A is an integral domain, then $\{[0]\}$ is prime in A/P. Hence these two chains are of the same length.

Definition 1.2.12. Let $Y \subset \mathbb{C}^n$ be an irreducible closed set. We define $\dim Y$ to be the supremum among all lengths l of chains $Y_l \subsetneq Y_{l-1} \subsetneq \cdots \subsetneq Y_1 \subsetneq Y_0 = Y$ of closed and irreducible subsets $Y_1, \ldots, Y_l$ contained in Y.

Remark 1.2.13. Chains $\cdots \subsetneq Y_l \cdots \subsetneq Y_1 \subsetneq Y_0$ of infinite length can not exist because this would imply infinite ascending chains of prime ideals that are not stable.

Remark 1.2.14. Note that if $Y_l \subsetneq Y_{l-1} \subsetneq \cdots \subsetneq Y_1 \subsetneq Y_0 = Y$, where Y_i's are irreducible and closed and thus I_{Y_i}'s are prime ideals, then we have

$$\begin{aligned}
& Y_l \subsetneq Y_{l-1} \subsetneq \cdots \subsetneq Y_1 \subsetneq Y_0 = Y \\
\iff & I_{Y_l} \supsetneq I_{Y_{l-1}} \supsetneq \cdots \supsetneq I_{Y_1} \supsetneq I_{Y_0} = I_Y \\
\iff & Z(I_{Y_l}) \subsetneq Z(I_{Y_{-1}}) \subsetneq \cdots \subsetneq Z(I_{Y_1}) \subsetneq Z(I_{Y_0}) = Z(I_Y) \\
\iff & \overline{Y_l} \subsetneq \overline{Y_{l-1}} \subsetneq \cdots \subsetneq \overline{Y_1} \subsetneq \overline{Y_0} = \overline{Y} \\
\iff & Y_l \subsetneq Y_{l-1} \subsetneq \cdots \subsetneq Y_1 \subsetneq Y_0 = Y.
\end{aligned}$$

By Proposition 1.2.3, a chain whose ideals are in $\mathrm{Spec}(\frac{\mathbb{C}[x_1,x_2,\ldots,x_n]}{I_Y})$ gives rise to a chain with the same length whose ideals are in $\mathrm{Spec}(I_Y)$, which implies $\dim Y \geq \dim(\frac{\mathbb{C}[x_1,x_2,\ldots,x_n]}{I_Y})$. By Remark 1.2.11, on the other hand, a chain whose ideals are in $\mathrm{Spec}(I_Y)$ gives rise to a chain with the same length whose ideals are in $\mathrm{Spec}(\frac{\mathbb{C}[x_1,x_2,\ldots,x_n]}{I_Y})$, which implies $\dim Y \leq \dim(\frac{\mathbb{C}[x_1,x_2,\ldots,x_n]}{I_Y})$.

Definition 1.2.15. Let Y be a closed set of $\mathbb{C}^n$. We define $\dim Y$ to be the maximum dimension $\max_{i=1,2,\ldots,s} Y_i$, where $Y_1, \ldots, Y_s$ are the unique irreducible components of Y.

Proposition 1.2.16. *Let Y be a closed set in $\mathbb{C}^n$ and $Y_1, Y_2, \ldots, Y_s$ the unique irreducible components of Y. Then*

$$\dim(\frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{I_Y}) = \max_{i=1,2,\ldots,s} \dim(\frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{I_{Y_i}}).$$

Proof. A chain of prime ideals containing I_{Y_i} can contain I_Y since $I_Y \subset I_{Y_i}$ for $i = 1, 2, \ldots, s$. Hence

$$\dim(\frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{I_Y}) \leq \max_{i=1,2,\ldots,s} \dim(\frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{I_{Y_i}}).$$

On the other hand, let $P \in \mathrm{Spec}(\mathbb{C}[x_1, x_2, \ldots, x_n])$ such that $I_Y = I_{Y_1} \cap I_{Y_2} \cap \cdots \cap I_{Y_s} \subset P$. Then we have $I_{Y_j} \subset P$ for some j, which implies

$$\dim(\frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{I_Y}) \geq \max_{i=1,2,\ldots,s} \dim(\frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{I_{Y_i}})$$

(we can replace I_Y by I_{Y_j} in a chain where every prime ideal contains I_Y). $\square$

Proposition 1.2.17. *Let J be an ideal of $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Then*

$$\sqrt{J} = \cap_{\substack{P \in \mathrm{Spec}(\mathbb{C}[x_1,x_2,\ldots,x_n]) \\ P \supset J}} P.$$

Moreover, there are finitely many factors in this intersection.

Proof. Let $Y_1, Y_2, \ldots, Y_s$ be the unique irreducible decomposition of $Z(J)$. Then we have

$$\begin{aligned} Z(J) &= Y_1 \cup Y_2 \cup \cdots \cup Y_s \\ \Rightarrow \sqrt{J} &= I_{Z(J)} = I_{Y_1} \cap I_{Y_2} \cap \cdots \cap I_{Y_s}. \end{aligned}$$

□

Proposition 1.2.18. *Let J be an ideal of $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Then*

$$\dim(\frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{J}) = \dim(\frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{\sqrt{J}}).$$

Proof. $\dim(\frac{\mathbb{C}[x_1,x_2,\ldots,x_n]}{J}) \geq \dim(\frac{\mathbb{C}[x_1,x_2,\ldots,x_n]}{\sqrt{J}})$ since $J \subset \sqrt{J}$. On the other hand, let $P \in \mathrm{Spec}(\mathbb{C}[x_1, x_2, \ldots, x_n])$ such that $P \supset J$. It is enough to show that $P \supset \sqrt{J}$. For $r \in \sqrt{J}$, we have $r^k \in J \subset P \Rightarrow r \in P$. □

The aim is to achieve three goals in the next few lectures:

- prove that $\dim \mathbb{F}[x_1, x_2, \ldots, x_n] = n$,
- Noether Normalization, and
- Hilbert Functions.

Definition 1.2.19 (integral)**.** Let B be a ring and A a subring of B. An element $b \in B$ is called *integral* over A if

$$b^n + \alpha_{n-1} b^{n-1} + \cdots + \alpha_1 b + \alpha_0 = 0$$

for some $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in A$.

Proposition 1.2.20 (AM\2.4)**.** *Let M be a finitely generated A-module, let I be an ideal of A, and let f be an A-module endomorphism of M such that $f(M) \subset IM = \{\alpha_1 w_1 + \cdots + \alpha_n w_n : n \in \mathbb{N}, \alpha_i \in I, w_i \in M\}$. Then f satisfies an equation of the form*

$$f^n + \alpha_1 f^{n-1} + \cdots + a_n = 0,$$

where α_i's are in I.

Proof. Please read the pictures or the book for the proof. □

Definition 1.2.21 (faithful module)**.** Let A be a ring and M an A-module. Then M is called *faithful* if there is no nonzero element α in A such that $\alpha M = 0$.

Proposition 1.2.22 (5.1\AM)**.** *Let B be a ring, let A be a subring of B, and let $b \in B$. Then the following are equivalent.*

1. *b is integral over A.*
2. *$A[b]$ is a finitely generated A-module.*
3. *$A[b]$ is contained in a ring C such that C is a finitely generated A-module.*
4. *there is a faithful $A[b]$-module M which is finitely generated over A.*

Proof.

- $1 \Rightarrow 2$). Since b is integral over A, we have

  $$b^n + \alpha_{n-1}b^{n-1} + \cdots + \alpha_1 b + \alpha_0 = 0 \iff b^n = -(\alpha_{n-1}b^{n-1} + \cdots + \alpha_1 b + \alpha_0)$$

  for some $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in A$, which implies $A[b]$ is (finitely) generated by $1, b, \ldots, b^{n-1}$.

- $2 \Rightarrow 3$). Take $C = A[b]$.

- $3 \Rightarrow 4$). Take $M = C$, which is a $A[b]$-module since $A = A[1] \subset A[b] \subset C$. Then M is faithful because if there is $p(b) \in A[b]$ such that $p(b)M = p(b) = C = 0$, then $p(1) \cdot 1 = 0$.

- $4 \Rightarrow 1$). Let $f : M \to M$ be an endomorphism of multiplying b, i.e., $f(m) = bm$ for each $m \in M$. Then $f(M) = bM \subset AM$. By Proposition 1.2.20, we have

  $$(b^n + \alpha_{n-1}b^{n-1} + \cdots + \alpha_1 b + \alpha_0)M = 0.$$

  Since M is faithful, $b^n + \alpha_{n-1}b^{n-1} + \cdots + \alpha_1 b + \alpha_0 = 0$. This implies b is integral over A.

$\square$

SI112: Advanced Geometry	Spring 2018
Lecture 26 — May 24th, Thursday	
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2 Lecture 26

2.1 Overview of This Lecture

The goal of lecture 26 is to prove Theorem 2.2.13.

2.2 Proof of Things

Proposition 2.2.1. *Let P be a prime ideal of a ring R, then $S = R - P$ is a multiplicatively closed set (i.e., $s_1 s_2 \in S$ for each $s_1, s_2 \in S$), and $0 \notin S$.*

Proof. Left as an exercise. $\square$

Proposition 2.2.2. *Let R be a ring and P, Q prime ideals of R. Let $\phi : R \to R_P$ be a ring homomorphism that maps $r \in R$ to $\frac{r}{1} \in R_P$. If $Q \cap (R \backslash P) \neq \emptyset$, then $\phi(Q) R_P$ is not a prime ideal.*

Proof. Since $Q \cap (R\backslash P) \neq \emptyset$, let $q \in Q \cap (R\backslash P)$. Then $\frac{q}{1} \in \phi(Q), \frac{1}{q} \in R_P$ and thus $\frac{1}{1} \in \phi(Q)R_P$, which implies that $\phi(Q)R_P$ is not prime. $\square$

For the sake of simplicity, we will use QR_P to denote $\phi(Q)R_P$ in what follows.

Proposition 2.2.3. *Let R be a ring and P, Q prime ideals of R with $Q \cap (R\backslash P) = \emptyset$. Then Q is a prime ideal of R if and only if QR_P is a prime ideal of R_P.*

Proof. Note that QR_P is properly contained by R_P since $1 \notin QR_P$ ($1 \notin Q$ and $Q \cap (R\backslash P) = \emptyset$), and that the set QR_P is of the form

$$\{\frac{q}{t} : \ q \in Q \text{ and } t \in R\backslash P\}.$$

Then

- $\Rightarrow$). Let $\frac{q_1}{t_1}, \frac{q_2}{t_2} \in R_P$ be such that $\frac{q_1 q_2}{t_1 t_2} \in QR_P$, then we have $q_1 q_2 \in Q$ since $t_1 t_2 \in R\backslash P$, which means that either $q_1 \in Q$ or $q_2 \in Q$. Hence we have either $\frac{q_1}{t_1} \in QR_P$ or $\frac{q_2}{t_2} \in QR_P$.

- $\Leftarrow$). Let $q_1, q_2 \in R$ be such that $q_1q_2 \in Q$ and let $t_1, t_2 \in R\backslash P$. Then we have $\frac{q_1}{t_1}, \frac{q_2}{t_2} \in R_P$ and $\frac{q_1q_2}{t_1t_2} \in QR_P$. This implies either $\frac{q_1}{t_1} \in QR_P$ or $\frac{q_2}{t_2} \in QR_P$. Hence we have either $q_1 \in Q$ or $q_2 \in Q$. There is a much quicker way: $\phi^{-1}(QR_P) = Q$ is prime by Proposition 1.2.3.

□

Remark 2.2.4 (remark for Proposition 2.2.3). The occurrence of QR_P is weird. It is because $\phi(Q)$ is in general not an ideal, but $\phi(Q)R_P$ always is.

Proposition 2.2.5. *Let P be a prime ideal of a ring R and $S = R - P$, then the ring $R_P = S^{-1}R$ contains only one maximal ideal equal to PR_P.*

Proof. The elements $\frac{p}{s}$ with $p \in P$ form an ideal $m = PR_P$ in R_P. If $\frac{b}{t} \notin m$, then $b \notin P$, hence $b \in S$ and therefore, noticing $t \in S$, $\frac{b}{t}$ is a unit in R_P ($\frac{b}{t} \cdot \frac{t}{b} = 1$). It follows that if I is an ideal in R_P and $I \not\subset m$, then I contains a unit, say $\frac{b}{t}$, which implies I contains $1 = \frac{b}{t} \cdot \frac{t}{b} = 1$ and hence $I = R_P$. Thus m is the only maximal ideal in R_P. □

Definition 2.2.6. Let $\phi : A \to B$ be an injective ring homomorphism, an element $b \in B$ is called *integral over A via ϕ* if

$$b^n + \phi(\alpha_{n-1})b^{n-1} + \cdots + \phi(\alpha_1)b + \phi(\alpha_0) = 0$$

for some $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in A$. Moreover, we say that B is integral over A if for each $b \in B$, b is integral over A.

Corollary 2.2.7 (5.2\AM)**.** *Let $b_1, b_2, \ldots, b_n \in B$ be such that they are integral over A. Then the ring $A[b_1, b_2, \ldots, b_n]$ is a finitely generated A-module.*

Proof. The case $n = 1$ is part of Proposition 1.2.22. Suppose inductively it holds for the case $n - 1$, i.e., the ring $A_{n-1} = A[b_1, b_2, \ldots, b_{n-1}]$ is a finitely generated A-module. Since $A \subset A_{n-1}$ and b_n is integral over A, b_n is integral over A_{n-1}. Therefore $A[b_1, b_2, \ldots, b_n] = A_{n-1}[b_n]$ is a finitely generated A_{n-1}-module by the case $n = 1$. Then we have

$$A[b_1, b_2, \ldots, b_n] = A_{n-1} + A_{n-1}b_n + A_{n-1}b_n^2 + \cdots A_{n-1}b_n^m,$$

where

$$A_{n-1} = Ap_1(b_1, b_2, \ldots, b_{n-1}) + p_2(b_1, b_2, \ldots, b_{n-1}) + \cdots + p_k(b_1, b_2, \ldots, b_{n-1})$$

for some

$$p_1(b_1, b_2, \ldots, b_{n-1}), p_2(b_1, b_2, \ldots, b_{n-1}), \ldots, p_k(b_1, b_2, \ldots, b_{n-1}) \in A_{n-1},$$

which implies that $A[b_1, b_2, \ldots, b_n]$ is finitely generated. □

Proposition 2.2.8 (5.6\AM)**.** *Let $A \subset B$ be rings, B integral over A (via the inclusion mapping).*

1. *If Q is an prime ideal of B and $P = A \cap Q$, then B/Q is integral over A/P.*
2. *If S is a multiplicatively closed subset of A, then $S^{-1}B$ is integral over $S^{-1}A$.*

Proof.

1. Let $b \in B$. Then there exists $\alpha_0, \dots, \alpha_{n-1} \in A$ such that

$$b^n + \alpha_{n-1}b^{n-1} + \dots + \alpha_1 b + \alpha_0 = 0.$$

Let $\pi_P : A \to A/P, \pi_Q : B \to B/Q$ be the canonical homomorphisms, let $i : A \to B$ be the inclusion mapping. Then by First Isomorphism Theorem, there exists an injective ring homomorphism $i^* : A/P \to B/Q$ such that $\pi_Q i = i^* \pi_P$ (have a look at the diagram in the pictures). Then we have

$$\begin{aligned} &\pi_Q(b^n + \alpha_{n-1}b^{n-1} + \dots + \alpha_1 b + \alpha_0) = 0 \\ \iff &[b]^n + [\alpha_{n-1}][b]^{n-1} + \dots + [\alpha_1][b] + [\alpha_0] = 0. \end{aligned}$$

Noticing that $[a_i] \in i^*(A/P)$ for $i = 0, 1, \dots, n-1$ and i^* is injective, B/Q is integral over A/P via i^*.

2. First note that $S^{-1}A \subset S^{-1}B$. Let $b \in B, s \in S$ (hence $\frac{b}{s} \in S^{-1}B$). Then there exists $\alpha_0, \dots, \alpha_{n-1} \in A$ such that

$$b^n + \alpha_{n-1}b^{n-1} + \dots + \alpha_1 b + \alpha_0 = 0.$$

Let $\phi : B \to S^{-1}B$ be a ring homomorphism that maps $x \in B$ to $\frac{x}{1} \in S^{-1}B$. Then we have

$$\begin{aligned} &\phi(b^n + \alpha_{n-1}b^{n-1} + \dots + \alpha_1 b + \alpha_0) = 0 \\ \iff &\frac{b^n}{1} + \frac{\alpha_{n-1}}{1}\frac{b^{n-1}}{1} + \dots + \frac{\alpha_1}{1}\frac{b}{1} + \frac{\alpha_0}{1} = 0 \\ \iff &(\frac{b}{s})^n + \frac{\alpha_{n-1}}{s}(\frac{b}{s})^{n-1} + \dots + \frac{\alpha_1}{s^{n-1}}(\frac{b}{s}) + \frac{\alpha_0}{s^n} = 0. \end{aligned}$$

Noticing that $\frac{a_k}{s^{n-k}} \in S^{-1}A$ for $k = 1, 2, \dots, n-1$, $S^{-1}B$ is integral over $S^{-1}A$ (via the inclusion mapping).

$\square$

Proposition 2.2.9 (5.7\AM)**.** *Let $A \subset B$ be integral domains, B integral over A (via the inclusion mapping). Then A is a field if and only if B is a field.*

Proof.

1. $\Rightarrow$). Let $y \in B$ and $y \neq 0$. There exists $\alpha_0, \dots, \alpha_{n-1} \in A$ such that

$$y^n + \alpha_{n-1}y^{n-1} + \cdots + \alpha_1 y + \alpha_0 = 0.$$

   Without loss of generality suppose that n is minimal. If $\alpha_0 = 0$, then $y(y^{n-1} + \alpha_{n-1}y^{n-2} + \cdots + \alpha_1) = 0$, which implies $y^{n-1} + \alpha_{n-1}y^{n-2} + \cdots + \alpha_1$, contradicting to the minimality of n. Hence $\alpha_0 \neq 0$. Since A is a field, $a_0^{-1} \in A \subset B$. Then

$$y[-a_0^{-1}(y^{n-1} + \alpha_{n-1}y^{n-2} + \cdots + \alpha_1)] = 1.$$

   Hence $y^{-1} = -a_0^{-1}(y^{n-1} + \alpha_{n-1}y^{n-2} + \cdots + \alpha_1) \in B$.

2. $\Leftarrow$). Let $x \in A \subset B$ and $x \neq 0$. Then $x^{-1} \in B$. There exists $\beta_0, \dots, \beta_{n-1} \in A$ such that

$$(x^{-1})^m + \beta_{n-1}(x^{-1})^{m-1} + \cdots + \beta_1(x^{-1}) + \beta_0 = 0.$$

   Hence

$$x^{-1} = -(\beta_{m-1} + \cdots + \beta_1 x^{m-2} + \beta_0 x^{m-1}) \in A.$$

□

Corollary 2.2.10 (5.8\AM). *Let $A \subset B$ be rings, B integral over A (via the inclusion mapping). If Q is an prime ideal of B and $P = A \cap Q$, then Q is a maximal if and only if P is maximal.*

Proof. Since P and Q are prime, A/P and B/Q are integral domains. By Proposition 2.2.8, B/Q is integral over A/P. Then by Proposition 2.2.9, Q is maximal if and only if B/Q is a field if and only if A/P is a field if and only if P is maximal. □

Proposition 2.2.11 (5.9\AM). *Let $A \subset B$ be rings, B integral over A (via the inclusion mapping). If $Q, Q' \in \operatorname{Spec}(B)$ such that $Q' \subset Q$ and $Q' \cap A = Q \cap A = P \in \operatorname{Spec}(A)$. Then $Q' = Q$.*

Proof. Note that by Proposition 2.2.3, $QB_P, Q'B_P \in \operatorname{Spec}(B_P)$, and that $P \subset Q$, $PA_P \subset A_P$ and $A_P \subset B_P$, we have

$$PA_P \subset QB_P \cap A_P \subsetneq A_P$$

(if $QB_P \cap A_P = A_P \iff A_P \subset QB_P$, then $1 \in QB_P \iff QB_P = B_P$, contradicting that QB_P is prime). Similarly

$$PA_P \subset Q'B_P \cap A_P \subsetneq A_P.$$

But by Proposition 2.2.5, PA_P is the unique maximal ideal of A_P, hence

$$QB_P \cap A_P = Q'B_P \cap A_P = PA_P.$$

Hence, by Proposition 2.2.8 and Corollary 2.2.10, $QB_P = Q'B_P$. Let $\phi : B \to B_P$ be a ring homomorphism that maps $x \in B$ to $\frac{x}{1} \in B_P$. Then we have

$$Q = \phi^{-1}(QB_P) = \phi^{-1}(Q'B_P) = Q',$$

as desired. $\square$

Theorem 2.2.12 ("lying over", 5.10\AM)**.** *Let $A \subset B$ be rings, B integral over A (via the inclusion mapping). Then for each $P \in \operatorname{Spec}(A)$, there exists $Q \in \operatorname{Spec}(B)$ such that $Q \cap A = P$.*

Proof. Firstly, by Proposition 2.2.8, B_P is integral over A_P. Let $\phi_A : A \to A_P, \phi_B : B \to B_P$ be ring homomorphisms that maps $a \in A$ and $b \in B$ to $\frac{a}{1} \in A_P$ and $\frac{b}{1} \in B_P$ respectively. Note that B_P is not the zero ring, let Q' be a maximal ideal of B_P. Then by Corollary 2.2.10, $Q' \cap A_P$ is maximal and hence $Q' \cap A_P = PA_P$. By commutativity of the diagram (review the picture), $P = \phi_A^{-1}(PA_P) = \phi_A^{-1}(Q' \cap A_P) = \phi_B^{-1}(Q') \cap A$. Let $Q = \phi(B)^{-1}(Q')$, finishing the proof. $\square$

Theorem 2.2.13. *Let $A \subset B$ be rings, B integral over A (via the inclusion mapping). Then* $\dim A = \dim B$*.*

Proof.

- Let $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$ be a chain of prime ideals of B. Then $Q_0 \cap A \subset Q_1 \cap A \subset \cdots \subset Q_n \cap A$ is a chain of prime ideals of A. If $Q_i \cap A = Q_{i+1} \cap A$ for some i, then we have $Q_i = Q_{i+1}$ by Proposition 2.2.11, contradicting to the construction of the chain. Hence $\dim A \geq \dim B$.

- On the other hand, let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_m$ be a chain of prime ideals of A. By Theorem 2.2.12 there exists $Q_0 \in \operatorname{Spec}(B)$ such that $P_0 = A \cap Q_0$. Let $\pi_A : A \to A/P_0, \pi_B : B \to B/Q_0$ be canonical homomorphisms. By Proposition 2.2.8, B/Q_0 is integral over A/P_0 via an injective homomorphism i^* (see the diagram). By again Theorem 2.2.12, there is $\overline{Q_1} \in \operatorname{Spec}(B/Q_0)$ such that $(i^*)(\overline{Q_1}) = \pi_A(P_1)$. At the same time we have $Q_0 \subset Q_1$ by letting $Q_1 = \pi_B^{-1}(\overline{Q_1}) \in \operatorname{Spec}(B)$. Suppose $Q_0 = Q_1$, then by the injectivity of i^*,

  $$\pi_A(P_1) = (i^*)(\overline{Q_1}) = (i^*)(\overline{0}) = \overline{0},$$

  which is impossible. Hence $Q_0 \subsetneq Q_1$. Proceeding in a similar way we can construct a chain $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_m$ of prime ideals of B, showing that $\dim A \leq \dim B$.

$\square$